Int. J. Solids Structures, 1972, Vol. 8, pp. 1051 to 1055. Pergamon Press. Printed in Great Britain

ON THE FINITE DEFLECTIONS OF THIN BEAMS

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Abstract—The numerical solution of three problems of finite deflection of uniform beams using the Euler–Bernoulli law of bending are presented. The problems are the uniformly loaded cantilever, the uniformly loaded simply supported beam and the column under axial loading.

1. INTRODUCTION

IN THIS paper we present the numerical solution to three problems of finite deflection of uniform beams using the Euler–Bernoulli law of bending. We consider a cantilever with uniformly distributed load and the solution obtained is compared with an approximate solution due to Rohde [1]. The second problem considered is that of a simply supported beam with uniform loading and the third problem is that of the buckling of a beam under uniform axial loading. Approximate solutions to these problems have been obtained by various authors notably Rohde [1], Seames and Conway [2] and Iyengar and Rao [3].

The first two problems have been recently considered by Wang [4], but unfortunately his analysis is in error as he commutes the derivatives d/dx and d/ds. The relationship of Rohde's approximate solution of the cantilever problem to the exact solution is shown to be quite different to that indicated in Wang's paper. The approximate solution underestimates the maximum deflection by a maximum of up to $5\frac{1}{4}$ per cent in Wang's paper it apparently overestimates the deflection by that amount. The third problem is the classical one of a column buckling under its own weight first studied by Euler (see Truesdell [5]).

The three problems reduce to standard two point boundary value problems and are integrated using a fourth order Runge–Kutta procedure. The method is not restricted to uniform loading and can be used for any continuous loading.

2. THE CANTILEVER WITH UNIFORMLY DISTRIBUTED LOAD

We consider a cantilever with uniform cross-section and choose the origin of our co-ordinate system at the free end. The arc length and slope of the neutral axis are denoted by s and θ as shown in Fig. 1.

If D is the flexural rigidity of the beam and M the bending moment then the Euler–Bernoulli law of bending states that

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\frac{M}{D}. \tag{1}$$

If the beam is subject to a uniform loading intensity w then it follows that

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = -\frac{w}{D}s\cos\theta. \tag{2}$$

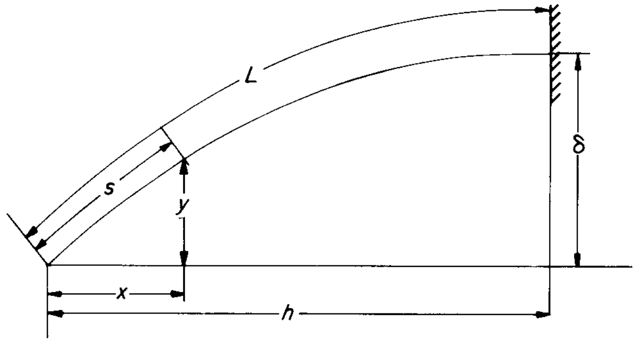

FIG. 1.

The boundary conditions are that

$$\frac{d\theta}{ds} = 0 \text{ at } s = 0 \quad \text{and} \quad \theta = 0 \text{ at } s = L,$$

where L is the length of the beam.

It is convenient to write $\bar{s} = s/L$ and to define a nondimensional parameter k by

$$k = \frac{wL^3}{D}. \tag{3}$$

The equation (2) is then

$$\frac{d^2\theta}{d\bar{s}^2} = -k\bar{s}\cos\theta \tag{4}$$

where $d\theta/d\bar{s} = 0$ at $\bar{s} = 0$ and $\theta = 0$ at $\bar{s} = 1$.

The second order equation (4) may now be written as a system of two first order equations and integrated by standard numerical methods giving θ as a function of $\bar{s}$. The x and y co-ordinates of points on the neutral axis are then given by

$$x(\bar{s}) = L\int_0^{\bar{s}} \cos\theta \, d\bar{s} \quad \text{and} \quad y(\bar{s}) = L\int_0^{\bar{s}} \sin\theta \, d\bar{s}. \tag{5}$$

The results of this paper were found using a fourth order Runge–Kutta procedure for the differential equations and Simpson's rule for the integrations [5]. The overall numerical accuracy obtained was 1×10^{-4} though there is no difficulty in obtaining any desired accuracy.

Figure 2 shows the variation of the maximum deflection δ with k, and the variation of the horizontal projection of the beam length, h with k. The dashed curves are the corresponding results taken as accurately as possible from (1). This shows that Rohde's method under-

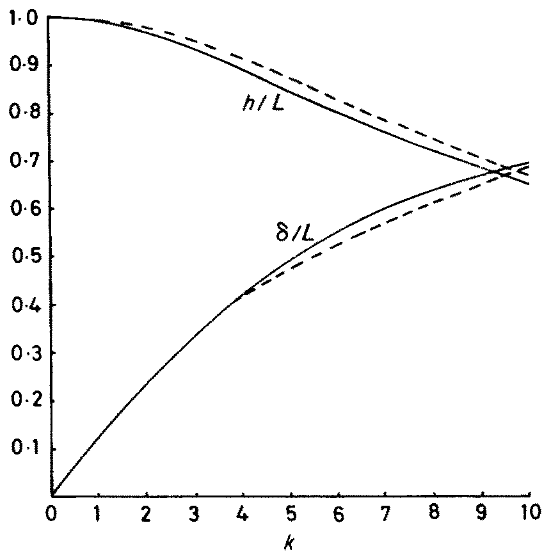

FIG. 2. Variation of δ/L and h/L with k for the cantilever.

estimated the maximum deflection by less than $5\frac{1}{4}$ per cent and in view of the fact that the Euler–Bernoulli theory is itself an approximate theory we may consider the approximate solution adequate.

3. THE SIMPLY SUPPORTED BEAM

We consider a uniform beam of length $2L$ simply supported at its ends a distance $2h$ apart. If we measure the arc length s $(=L\bar{s})$ from one end then we have

$$\frac{d^2\theta}{d\bar{s}^2} = -k(1-\bar{s})\cos\theta \tag{6}$$

where $d\theta/ds = 0$ at $\bar{s} = 0$ and $\theta = 0$ at $\bar{s} = 1$.

This problem is solved in a similar manner to the cantilever and the results are shown in Fig. 3.

4. THE BUCKLING OF A COLUMN UNDER AXIAL LOADING

As in Section 2 we measure s along the column from the free end and if θ is the angle the tangent at a point makes with the vertical we have

$$\frac{d^2\theta}{d\bar{s}^2} = -k\bar{s}\sin\theta \tag{7}$$

where $d\theta/d\bar{s} = 0$ at $\bar{s} = 0$ and $\theta = 0$ at $\bar{s} = 1$.

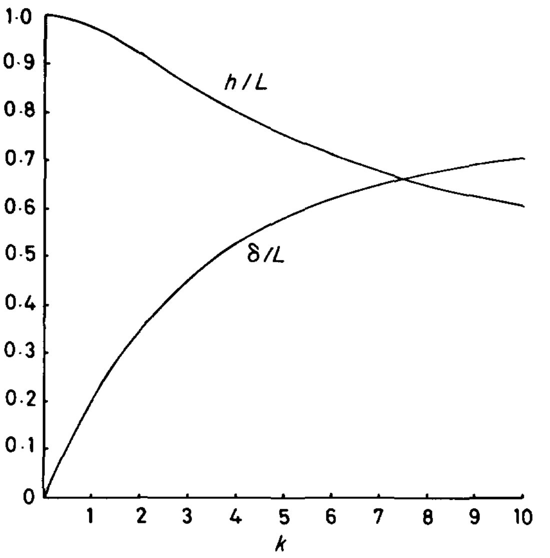

FIG. 3. Variation of δ/L and h/L with k for the simply supported beam.

The numerical results for this problem confirm the critical value for k of 7·837 obtained from the linear theory. The graph showing the variation of maximum horizontal displacement δ with k is shown on Fig. 4 and shows the well known phenomena that a small increase of k above the critical value initially produces a large displacement. Figure 4 also shows the variation of maximum height h with k and Fig. 5 shows the shape of the column for values of k up to 20. Figure 5 may be compared with the set of curves given in Timoshenko and Gere [6] for the light column with an isolated load at its upper end.

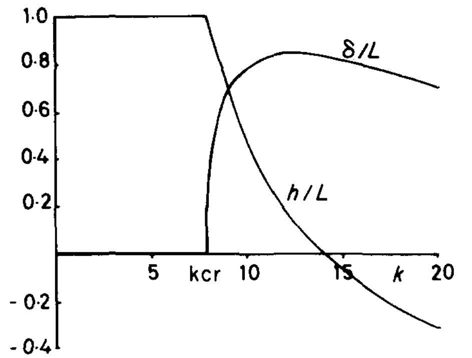

FIG. 4. Variation of δ/L and h/L with k for the column.

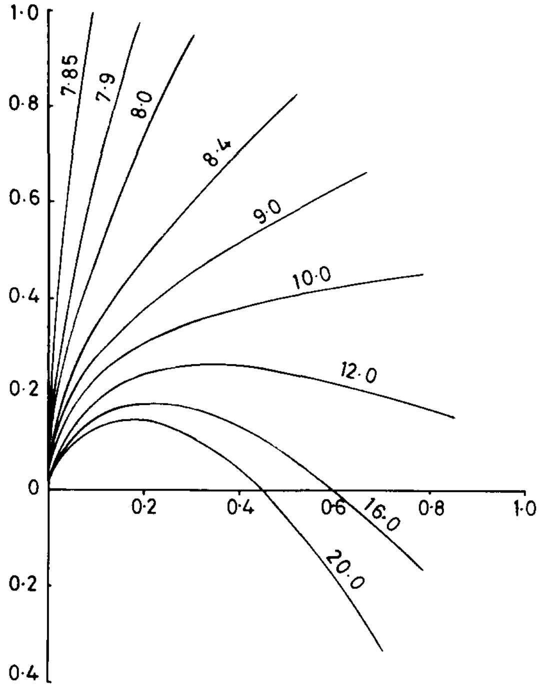

FIG. 5. Shape of column for values of k up to 20.

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(*Received* 7 *September* 1971; *revised* 17 *February* 1972)

Абстракт—Дается численное решение трех задач конечного прогиба для однородных бажок, используя закон Эйлера–Бернулли. Задачи касаются равномерно нагруженной консоли, равномерно нагруженной, свободно опертой балки и колонны под влиянием осевой нагрузки.